On the spectra of Schwarz matrices with certain sign patterns

Mikhail Tyaglov*
Technische Universität Berlin, Institut für Mathematik,
MA 4-5, Strasse des 17. Juni 136, 10623, Berlin, Germany
Email: tyaglov@math.tu-berlin.de

January 4, 2012

Key words. Schwarz matrices, inverse spectral problem, generalized Hurwitz polynomials, continued fractions, tridiagonal matrices.

AMS subject classification. 15A29, 47A75, 47B36, 26C10.

Abstract

The direct and inverse spectral problems are solved for a wide subclass of the class of Schwarz matrices. A connection between the Schwarz matrices and the so-called generalized Hurwitz polynomials is found. The known results due to H. Wall and O. Holtz are briefly reviewed and obtained as particular cases.

1 Introduction

In this work, we consider the matrices of the form

$$\begin{bmatrix} -b_0 & 1 & 0 & \dots & 0 & 0 \\ -b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -b_{n-1} & 0 \end{bmatrix}, \qquad b_k \in \mathbb{R}\backslash\{0\} \tag{1.1}$$

that usually called the *Schwarz matrices*[1]. We solve direct and inverse problems for such matrices with certain sign patterns.

These matrices are well-studied from the matrix theory point of view (see e.g. [2, 14, 3, 4, 5, 6] and references there). Here we use the method due to Wall [17, 18] to solve the inverse spectral problem and our results on the generalized Hurwitz polynomials to solve the direct spectral problem for the Schwarz matrices (1.1) with a wide class of sign patterns. The case of all b_k positive was considered by H. Wall [17] and later by H. Schwarz [15] and many other authors. The case of all b_k negative was considered by O. Holtz [12]. Here we use formulæ obtained by Wall in [17] which connect the entries of the matrix (1.1) with coefficients of its characteristic polynomial (see formulæ (2.8) below) to use the so-called generalized Hurwitz theorem established in [16].

*Supported by the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC grant agreement no. 259173.

[1]Schwarz himself [15] considered also the matrices whose $(1,1)$th entry is zero while (n,n)th entry is nonzero. Sometimes such matrices are called the Schwarz matrices as well (see e.g. [7]).

In Section 2, we review results due to Wall that was obtained in [17]. Section 3 is devoted to all solved direct and inverse problems for the Schwarz matrices. In Section 4, we recall some basic facts on the generalized Hurwitz polynomials established in [16]. In Section 5, we prove our main theorems on the direct and inverse problems for Schwarz matrices with certain sign patterns. Finally, in Section 6, we apply our results of Section 5 to matrices (1.1) with one sign change in the sequence $b_1, \dots, b_{n-1}$. In particular, we prove the direct and inverse problems for the matrices (1.1) with $b_1 > 0$, $b_2, \dots, b_{n-1} < 0$ which was considered in [1].

2 Wall's continued fractions and the Schwarz matrices

Given a monic real polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n, \tag{2.1}$$

we represent it as follows

$$p(z) = p_0(z^2) + z p_1(z^2),$$

where the polynomials $p_0(u)$ and $p_1(u)$ are the even and odd parts of the polynomial p, respectively:

$$p_0(u) = a_n + a_{n-2}u + a_{n-4}u^2 + \cdots, \tag{2.2}$$

$$p_1(u) = a_{n-1} + a_{n-3}u + a_{n-5}u^2 + \cdots \tag{2.3}$$

We also introduce the polynomial q as follows

$$q(z) = \begin{cases} p_0(z^2) & \text{if} \quad n = 2l+1, \\ z p_1(z^2) & \text{if} \quad n = 2l. \end{cases} \tag{2.4}$$

Let us associate with the polynomial p the following determinants called the *Hurwitz determinants*:

$$\Delta_j(p) = \det \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \dots & a_{2j-1} \\ 1 & a_2 & a_4 & a_6 & \dots & a_{2j-2} \\ 0 & a_1 & a_3 & a_5 & \dots & a_{2j-3} \\ 0 & 1 & a_2 & a_4 & \dots & a_{2j-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_j \end{bmatrix}, \quad j = 1, \dots, n, \tag{2.5}$$

where we set $a_i \equiv 0$ for $i > n$.

In 1945, H. Wall established [17] (see also [18]) the following theorem.

Theorem 2.1 (Wall). *If the coefficients of the polynomial p given in (2.1) satisfy the inequalities*

$$\Delta_j(p) \neq 0, \qquad j = 1, \dots, n, \tag{2.6}$$

then there is determined uniquely a continued fraction of the form

$$\frac{q(z)}{p(z)} = \cfrac{b_0}{z + b_0 + \cfrac{b_1}{z + \cfrac{b_2}{\ddots + \cfrac{b_{n-1}}{z}}}} \tag{2.7}$$

where q is defined in (2.4) and the real coefficients b_k are given by the formulæ

$$\begin{aligned} b_0 &= \Delta_1(p), \\ b_k &= \frac{\Delta_{k-2}(p)\Delta_{k+1}(p)}{\Delta_{k-1}(p)\Delta_k(p)}, \qquad k = 1, \dots, n-1, \end{aligned} \tag{2.8}$$

where $\Delta_{-1}(p) = \Delta_0(p) \equiv 1$.

Conversely, the coefficients in the last denominator of a continued fraction of the form (2.7) satisfy the inequalities (2.6).

From the form of the continued fraction (2.7) it is easy to see that

$$\frac{q(z)}{p(z)} = b_0\,((zE_n - J_n)^{-1}e_1, e_1),$$

where e_1 is the first coordinate vector in $\mathbb{R}^n$, E_n is the $n \times n$ unity matrix, and

$$J_n = \begin{bmatrix} -b_0 & 1 & 0 & \dots & 0 & 0 \\ -b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -b_{n-1} & 0 \end{bmatrix}, \tag{2.9}$$

where the nonzero real entries b_k are exactly the coefficients of the continued fraction (2.7). In other words, the polynomial $p(z)$ is the characteristic polynomial of the matrix J_n, while the polynomial $q(z)/b_0$ is the characteristic polynomial of the principal submatrix of the matrix J_n obtained by deleting the first column and the first row. Thus, we come to the following conclusion.

Theorem 2.2. *The characteristic polynomial* p *of the matrix* J_n *defined in (2.9) satisfies the inequalities (2.6). Conversely, for every real polynomial* p *satisfying the inequalities (2.6), there exists a unique matrix of the form (2.9) whose characteristic polynomial is* p.

The matrices of the form (2.9) are called the *Schwarz matrices* after H. Schwarz[2] who developed a method of transformation a given nonderogatory matrix with the characteristic polynomial satisfying (2.6) to the form (2.9) (see [15]).

Theorem 2.2 provides a solution of somewhat direct and inverse problems for tridiagonal matrices of the form (2.9). These problems, however, are not spectral and concern properties of the characteristic polynomial of J_n. Nevertheless, their solution is important to solving spectral direct and especially inverse problems for the matrices of the form (2.9). Thus it makes sense to give a bit more detailed explanation of Theorem 2.2.

So given a polynomial p defined by (2.1) and satisfying the inequalities (2.6), the matrix J_n such that $p(z) = \det(zE_n - J_n)$ can be reconstruct, for instance, by formulæ (2.8). However, one can also run a Sturm algorithm as it was noted in [15].

Indeed, let

$$f_0(z) := p(z) \qquad \text{and} \qquad f_1(z) := \frac{q(z)}{b_0},$$

where $q(z)$ is defined in (2.4). The polynomials f_0 and f_1 are monic, and f_1 is even or odd by construction. Now we construct a sequence of monic polynomials f_k, $\deg f_k = n - k$, by the following process

$$\begin{gathered} b_1 f_2(z) := f_0(z) - (z + b_0) f_1(z), \\ b_2 f_3(z) := f_1(z) - z f_2(z), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ b_{n-2} f_{n-1}(z) := f_{n-3}(z) - z f_{n-2}(z), \\ b_{n-1} := f_{n-2}(z) - z f_{n-1}(z). \end{gathered}$$

Thus, these equations give us all the entries b_k of the matrix J_n in (2.9). Moreover, the polynomials f_k, $k = 1, \dots, n$, are even or odd, and $f_k(z)$ is the characteristic polynomial of the principal submatrix of J_n obtained by deleting first k rows and first k columns.

[2]Schwarz considered matrices (n, n)th nonzero entries rather than $(1, 1)$th as we do. In this work, we follow H.Wall who considered matrices (2.9) earlier than Schwarz.

We, finally, investigate the structure of the matrix (2.9) in detail. Let again p be its characteristic polynomial: $p(z) = \det(zE_n - J_n)$. Consider the following auxiliary matrix

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -b_{n-1} & 0 \end{bmatrix}$$

and its submatrix

$$A_{n-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -b_2 & 0 & 1 & \dots & 0 & 0 \\ 0 & -b_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -b_{n-1} & 0 \end{bmatrix}$$

obtained from A_n by deleting its first row and column. It is easy to see that

$$p(z) = \det(zE_n - J_n) = \det(zE_n - A_n) + b_0 \det(zE_{n-1} - A_{n-1})$$

It is also clear that if $p(z) = p_0(z^2) + zp_1(z^2)$, where $p_0(u)$ and $p_1(u)$ are the even and odd parts of p, respectively, then
for $n = 2l$,

$$p_0(z^2) = \det(zE_n - A_n) \quad \text{and} \quad zp_1(z^2) = b_0 \det(zE_{n-1} - A_{n-1}) \tag{2.10}$$

for $n = 2l + 1$,

$$zp_1(z^2) = \det(zE_n - A_n) \quad \text{and} \quad p_0(z^2) = b_0 \det(zE_{n-1} - A_{n-1})$$

These formulæ imply the following simple fact.

Proposition 2.3. *Let the polynomial $p(z) = p_0(z^2) + zp_1(z^2)$ be the characteristic polynomial of the matrix J_n given in (2.9), $p(z) = \det(zE_n - J_n)$. Then the polynomial $q(z) = (-1)^{\left[\frac{n+1}{2}\right]}[p_0(-z^2) + (-1)^n zp_1(-z^2)]$ is the characteristic polynomial of the matrix*

$$\begin{bmatrix} b_0 & 1 & 0 & \dots & 0 & 0 \\ b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & b_{n-1} & 0 \end{bmatrix} \tag{2.11}$$

Proof. We prove the proposition for $n = 2l$. For $n = 2l + 1$, it can be established analogously.

So let $n = 2l$. Then $\left[\frac{n+1}{2}\right] = l$, and the polynomial q has the form

$$q(z) = (-1)^l p_0(-z^2) + (-1)^l zp_1(-z^2)$$

Using formulæ (2.10) one can obtain

$$(-1)^l p_0(-z^2) = (-1)^l \det(-izE_n - A_n) = \det(zE_n - iA_n) = \det(zE_n - B_n),$$

where

$$B_n = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & b_{n-1} & 0 \end{bmatrix}.$$

Here we used the fact (see e.g. [9, Chapter II]) that the characteristic polynomial of any tridiagonal matrix does not depend on $(i+1,i)$th and $(i,i+1)$th entries separately but on their product, so the matrices iA_n and B_n have the same characteristic polynomial.

Analogously we have

$$(-1)^l zp_1(-z^2) = i(-1)^l b_0 \det(-izE_n - A_{n-1}) = \\ -b_0 \det(zE_n - iA_{n-1}) = -b_0 \det(zE_n - B_{n-1}),$$

where

$$B_{n-1} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ b_2 & 0 & 1 & \dots & 0 & 0 \\ 0 & b_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & b_{n-1} & 0 \end{bmatrix}.$$

Thus, we get

$$q(z) = \det(zE_n - B_n) - b_0 \det(zE_n - B_{n-1}),$$

so $q(z)$ is the characteristic polynomial of the matrix (2.11). □

3 Some solved direct and inverse spectral problems for the Schwarz matrices

In the previous section, we described properties of the characteristic polynomials of the Schwarz matrices of the form (2.9) and recall methods of reconstruction such matrices from their characteristic polynomials. However, we are interested in direct and inverse *spectral* problems of the Schwarz matrices.

It is natural to study a dependence of the spectrum of the matrix J_n given in (2.9) in terms of *signs* of the entries b_k of this matrix. Since we have the relations (2.8) between the entries b_k of the matrix J_n and the coefficients a_j of its characteristic polynomials, it makes sense to use the results of the theory of root location of polynomials which use signs of Hurwitz minors.

The most known such result is the Hurwitz theorem stating that a real polynomial $p(z)$ given by (2.1) has all its zeroes in the open left half-plane of the complex plane if and only if its Hurwitz minors (2.5) are *positive*. We recall that a real polynomial is called *Hurwitz stable* if all its zeroes lie in the open left half-plane.

Using the Hurwitz theorem H. Wall established the following fact in [17, p.314] (see also [15]).

Theorem 3.1 (Wall)**.** *The Schwarz matrix J_n given in (2.9) has all its eigenvalues in the open left half-plane if all the entries b_k are positive. Conversely, given a sequence of complex numbers $\lambda_1, \dots, \lambda_n$ with negative real parts, there exists a unique Schwarz matrix J_n of the form (2.9) with $b_k > 0$, $k = 0, 1, \dots, n-1$, such that*[3] $\sigma(J_n) = \{\lambda_1, \dots, \lambda_n\}$.

Thus this theorem solves the direct and inverse spectral problems for stable Schwarz matrices, that is, the Schwarz matrices with positive b_k, that are sometimes called *Routh canonical forms* (see e.g. [14, 4]). Note that Wall's work [17] where he considered Hurwitz stable polynomials as characteristic polynomials of matrices (2.9) with positive b_k appeared earlier than the paper [15] by H. Schwarz.

Next result regarding eigenvalue location of the matrix (2.9) is based on the so-called Routh-Hurwitz theorem established by Gantmacher in [11, Theorem 4, p. 230].

[3] $\sigma(J_n)$ denotes the spectrum of the matrix J_n.

Theorem 3.2 (Routh-Hurwitz). *Let the polynomial p be defined in (2.1) and satisfy (2.6). The number m of roots of p which lie in the open right half-plane is given by the formula*

$$m = v\left(1, \Delta_1(p), \frac{\Delta_2(p)}{\Delta_1(p)}, \frac{\Delta_3(p)}{\Delta_2(p)}, \ldots, \frac{\Delta_n(p)}{\Delta_{n-1}(p)}\right)$$

or equivalently

$$m = v(1, \Delta_1(p), \Delta_3(p), \ldots) + v(1, \Delta_2(p), \Delta_4(p), \ldots),$$

where $\Delta_j(p)$ are the Hurwitz determinants of p, and $v(c_0, c_1, \ldots, c_l)$ denotes the number of sign changes in the sequence $[c_0, c_1, \ldots, c_l]$.

Remark 3.3. Note that in Theorem 3.2, all Hurwitz determinants of the polynomial p are nonzero by assumption, so we use the standard calculation of the sign changes in the sequences of the Hurwitz determinants. However, this theorem is also true in the case when some of Hurwitz determinants of p equal zero [11, §8, p. 235] (see also [13] or comments to Theorem 4.3 on the page 8).

Using Theorem 3.2 and formulæ (2.8) one can easily obtain the following result due to Schwarz [15, Satz 5] which also follows from Theorem D and formulæ (2.1) of Wall's work [17].

Theorem 3.4. *Given a real matrix J_n as in (2.9), the number of negative terms in the sequence*

$$b_0,\ b_0b_1,\ b_0b_1b_2, \ldots,\ b_0b_1\cdots b_{n-1} \tag{3.1}$$

is equal to the number of eigenvalues of J_n in the open right half-plane of the complex plane.

This theorem uses sign patterns of the entries of the matrix J_n to localize distributions of its eigenvalues. So this result can be identified as the solution of the direct spectral problem of the matrix J_n. The inverse spectral problem is somewhat trivial in light of Theorems 2.2 and 3.2 and formulæ (2.8).

Theorem 3.5. *Let $\lambda_1, \ldots, \lambda_n$ be a sequence of complex numbers with m numbers in the open right half-plane and $n-m$ numbers in the open left half-plane such that the polynomial $p(z) = \prod\limits_{i=1}^{n}(z-\lambda_i)$ satisfies the inequalities (2.6). There exists a unique matrix J_n of the form (2.9) such that the number of negative terms in the sequence (3.1) constructed with the entries of J_n equals m and $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$.*

Note that in Theorem 3.1 we did not need to suppose the polynomial $p(z) = \prod\limits_{i=1}^{n}(z-\lambda_i)$ to satisfy (2.6), because all Hurwitz stable polynomials automatically satisfy them by the Hurwitz theorem we mentioned above (see also Theorem 3.2 and remark after it).

Theorem 3.1 deals with the Schwarz matrices with positive b_k, so it is natural to study the Schwarz matrices (2.9) with all negative b_k. This problem was solved by O. Holtz in [12, Corollary 2], where she proved the following[4].

Theorem 3.6. *Let the matrix J_n be defined in (2.9) with $b_k < 0$, $k = 0, \ldots, n-1$. Then its eigenvalues λ_i are simple real and satisfy the inequalities*

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1}\lambda_n > 0. \tag{3.2}$$

Conversely, for any sequence of real numbers $\lambda_1, \ldots, \lambda_n$ distributed as in (3.2), there exists a unique matrix J_n of the form (2.9) with $b_k < 0$, $k = 0, \ldots, n-1$, such that $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$.

[4]It is worth to note that there is a mistake in the proof of the main theorem, Theorem 1, in [12]. However, this mistake can be easily corrected, while the statement of the theorem is valid.

This theorem was proved in [1] by a technique different from one used in [12]. However, it can be proved easily using properties of generalized Hurwitz polynomials [16] (see Remark 4.9). We just note that in Theorem 3.6, there is no requirement for the polynomial $p(z) = \prod_{i=1}^{n}(z-\lambda_i)$ to satisfy the inequalities (2.6). As we will show, the polynomials with the distribution of zeroes as in (3.2) automatically satisfy (2.6), since they are dual (in some sense[5]) to Hurwitz stable polynomials.

Finally, we should mention that in [1], there was an attempt to solve direct and inverse problems for the matrix (2.9) with $b_0 < 0$, $b_1 > 0$ and $b_k < 0$ for $k = 2, \dots, n-1$. However, their result is incorrect. We include the correct version of their Theorem 9 as an example of application of our results (see Theorem 6.5).

4 Generalized Hurwitz polynomials, basic properties

In this section, we define (almost) generalized Hurwitz polynomials [16] and review their basic property, which will be helpful to study spectral problems of the Schwarz matrices.

Definition 4.1. A real polynomial p is called *generalized Hurwitz polynomial* of type I of order $\varkappa$, where[6] $1 \leqslant \varkappa \leqslant \left[\frac{n+1}{2}\right]$, if it has exactly $\varkappa$ zeroes in the *closed* right half-plane, all of which are nonnegative and simple:

$$0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_\varkappa,$$

such that $p(-\mu_i) \neq 0$, $i = 2, \dots, \varkappa$, $p(-\mu_1) \neq 0$ if $\mu_1 > 0$, and p has an odd number of zeroes, counting multiplicities, on each interval $(-\mu_\varkappa, -\mu_{\varkappa-1}), \dots, (-\mu_3, -\mu_2)$, $(-\mu_2, -\mu_1)$. Moreover, the number of zeroes of p on the interval $(-\mu_1, 0)$ (if any) is even, counting multiplicities. The other *real* zeroes lie on the interval $(-\infty, -\mu_\varkappa)$: an odd number of zeroes, counting multiplicities, when $n = 2l$, and an even number of zeroes, counting multiplicities, when $n = 2l+1$. All nonreal zeroes of p (if any) are located in the open left half-plane of the complex plane.

Thus, the order $\varkappa$ of a generalized Hurwitz polynomial of type I indicates the number of its zeroes in the closed right half-plane. Moreover, the zeroes of a generalized Hurwitz polynomial in the closed right half-plane structure the distribution of its negative zeroes, so not every real polynomial with only real simple zeroes in the closed right half-plane is generalized Hurwitz. The generalized Hurwitz polynomials of type I of order 0 are obviously Hurwitz stable polynomials, since they have no zeroes in the open right half-plane.

The generalized Hurwitz polynomials of type II is a generalization of real polynomials with zeroes in the open right half plane.

Definition 4.2. A real polynomial $p(z)$ is generalized Hurwitz of type II if the polynomial $p(-z)$ is generalized Hurwitz of type I.

It is clear that all results obtained for the generalized Hurwitz polynomials of type I can be easily reformulated for the generalized Hurwitz polynomials of type II. Thus we formulate all results in this section only for generalized Hurwitz polynomials of type I.

The main fact about generalized Hurwitz polynomials we use in this paper is the following theorem established in [16].

Theorem 4.3 (Generalized Hurwitz theorem)**.** *The polynomial p given in (2.1) is generalized Hurwitz if and only if*

$$\Delta_{n-1}(p) > 0,\ \Delta_{n-3}(p) > 0,\ \Delta_{n-5}(p) > 0, \dots \tag{4.1}$$

[5] See Theorem 4.6.

[6] Here $[\alpha]$ denotes the maximal integer not exceeding α.

The order $\varkappa$ of the polynomial p equals

$$\varkappa = \mathrm{V^F}(\Delta_n(p), \Delta_{n-2}(p), \ldots, 1) \qquad \text{if} \quad p(0) \neq 0, \tag{4.2}$$

or

$$\varkappa = \mathrm{V^F}(\Delta_{n-2}(p), \Delta_{n-4}(p), \ldots, 1) + 1 \qquad \text{if} \quad p(0) = 0, \tag{4.3}$$

where $V^F(c_1, \ldots, c_n)$ denotes the number of sign changes in the sequence $\{c_1, \ldots, c_n\}$ calculated in accordance with the Frobenius rule of signs.

Recall that the Frobenius rule of signs is the following.

Frobenius rule of signs [8] (see also [10, Ch. X, §10] and [13, Ch. 2]). *Given a sequence of real numbers $\{c_1, \ldots, c_n\}$, where $c_1 c_n \neq 0$, if, for some i and j $(0 \leqslant i \leqslant j)$,*

$$c_i \neq 0, \quad c_{i+1} = c_{i+2} = \cdots = c_{i+j} = 0, \quad c_{i+j+1} \neq 0$$

then the number $V^F(c_1, \ldots, c_n)$ of Frobenius sign changes must be calculated by assigning signs as follows:

$$\operatorname{sign} c_{i+\nu} = (-1)^{\frac{\nu(\nu-1)}{2}} \operatorname{sign} c_i, \quad \nu = 1, 2, \ldots, j.$$

The Frobenius rule of signs was introduced by Frobenius [8] for calculating the number of sign changes in a sequence of Hankel minors. For details, see [13].

Since we consider only polynomials with nonzero Hurwitz determinants in this work, in the rest of the paper the number of Frobenius sign changes V^F will be changed by the standard number of sign changes v, and the formula (4.3) will not be used, since $\Delta_n(p) = 0$ if $p(0) = 0$.

By (4.2)–(4.3), $\varkappa = 0$ if and only if $\Delta_{n-2k}(p) > 0$, $k = 0, 1, \ldots, \left[\frac{n-1}{2}\right]$. As we mentioned above, the generalized Hurwitz polynomials with $\varkappa = 0$ are Hurwitz stable polynomials. Thus, Theorem 4.3 implies that a real polynomial p of degree n is Hurwitz stable if and only if $\Delta_j(p) > 0$, $j = 1, \ldots, n$. This is exactly the Hurwitz stability criterion.

On the other side, the formulæ (4.2)–(4.3) imply that $\varkappa = \left[\frac{n+1}{2}\right]$ with $p(0) \neq 0$ if and only if

$$(-1)^d \Delta_n(p) > 0, (-1)^d \Delta_{n-2}(p) > 0, \ldots, \quad \text{where} \quad d = \left[\frac{n+1}{2}\right]. \tag{4.4}$$

In this case, the generalized Hurwitz polynomial p of type I has neither nonreal nor multiple zeroes, so its zeroes are real and simple. Moreover, they are distributed as follows:

$$0 < \lambda_1 < -\lambda_2 < \lambda_3 < \ldots < (-1)^{n-1}\lambda_n. \tag{4.5}$$

Definition 4.4. A real polynomial whose zeroes are distributed as in (4.5) is called *self-interlacing* of type I.

Analogously to the general case, we introduce the self-interlacing polynomials of type II.

Definition 4.5. A polynomial $p(z)$ is called self-interlacing of type II if $p(-z)$ is self-interlacing of type I, or equivalently if its zeroes are distributed as follows:

$$0 < -\lambda_1 < \lambda_2 < -\lambda_3 < \ldots < (-1)^n \lambda_n. \tag{4.6}$$

From Definitions 4.4–4.5 it is easy to see that a real polynomial $p(z)$ is self-interlacing (of type I or II) if and only if it has real and simple zeroes which interlace the zeroes of the polynomial $p(-z)$.

If we return now to Theorem 3.6, we will see from (3.2) that the characteristic polynomials of the matrices (2.9) with all $b_k < 0$ are self-interlacing polynomials: of type I for odd n, and of type II for even n.

In [16], there was also established the following important fact about the relation (in fact, duality) between Hurwitz stable and self-interlacing polynomials. We will use this fact later to reveal a relation between Theorems 3.1 and 3.6 (see Remark 4.9).

Theorem 4.6. *A polynomial $p(z) = p_0(z^2) + zp_1(z^2)$ is self-interlacing of type I if and only if the polynomial $q(z) = p_0(-z^2) - zp_1(-z^2)$ is Hurwitz stable, where $p_0(u)$ and $p_1(u)$ are the even and odd parts of p, respectively (see (2.2)–(2.3)).*

Indeed, there can be established a more general fact.

Theorem 4.7 ([16])**.** *A polynomial $p(z) = p_0(z^2) + zp_1(z^2)$, $p(0) \neq 0$, is generalized Hurwitz of order $\varkappa$ of type I (type II) if and only if the polynomial $q(z) = p_0(-z^2) - zp_1(-z^2)$ is generalized Hurwitz of order $\left[\frac{n+1}{2}\right] - \varkappa$ of type I (type II), where $n = \deg p$.*

Remark 4.8. *From Definition 4.2 and Theorem 4.7 it is clear that if $p(z) = p_0(z^2) + zp_1(z^2)$ is generalized Hurwitz of type I, then $p(z) = p_0(-z^2) + zp_1(-z^2)$ is generalized Hurwitz of type II.*

Remark 4.9. *Theorem 3.1, Proposition 2.3, Theorem 4.6 and Remark 4.8 imply Theorem 3.6. Conversely, Wall's Theorem 3.1 can be obtained from Theorem 3.6, Theorem 4.6, and Proposition 2.3 taking into account Remark 4.8.*

Finally, let us introduce the so-called almost generalized Hurwitz polynomials.

Definition 4.10. A real polynomial $p(z)$ called almost generalized Hurwitz of order $\varkappa$ of type I (type II) if the polynomial $zp(z)$ is generalized Hurwitz of order $\varkappa + 1$ of type I (resp. type II).

Remark 4.11. *Note that any almost generalized Hurwitz polynomial of order 0 of type I is a Hurwitz stable polynomial, while any almost generalized Hurwitz polynomial of type I of degree $2l$ and of order l is a self-interlacing polynomial of type II. Also any almost generalized Hurwitz polynomial of type II of degree $2l+1$ and of order l is a self-interlacing polynomial of type I.*

For almost generalized Hurwitz polynomials we have the following basic theorem analogous to Theorem 4.3 (see [16]).

Theorem 4.12. *The polynomial p given in (2.1) is generalized Hurwitz if and only if*

$$\Delta_n(p) > 0,\ \Delta_{n-2}(p) > 0,\ \Delta_{n-4}(p) > 0, \dots \tag{4.7}$$

The order $\varkappa$ of the polynomial p equals

$$\varkappa = \mathrm{V}^{\mathrm{F}}(\Delta_{n-1}(p), \Delta_{n-3}(p), \dots, 1). \tag{4.8}$$

where $V^F(c_1, \dots, c_n)$ denotes the number of sign changes in the sequence $\{c_1, \dots, c_n\}$ calculated in accordance with the Frobenius rule of signs.

Note that almost generalized Hurwitz polynomials do not vanish at zero, so they order equal the number of positive simple zeroes. One can easily describe the distribution of zeroes of almost generalized Hurwitz polynomials from Definitions 4.1 and 4.10. Moreover, if a real polynomial is generalized Hurwitz and almost generalized Hurwitz simultaneously, then it is Hurwitz stable.

5 Direct and inverse spectral problems for Schwarz matrices

Let us again consider the Schwarz matrix

$$J_n = \begin{bmatrix} -b_0 & 1 & 0 & \dots & 0 & 0 \\ -b_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -b_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -b_{n-1} & 0 \end{bmatrix} \tag{5.1}$$

with all b_k nonzero, and denote by $p(z)$ its characteristic polynomial, that is, $p(z) = \det(zE_n - J_n)$. From formulæ (2.8) it is easy to get the following

$$b_0 = \Delta_1(p), \quad b_{2j-1}b_{2j} = \frac{\Delta_{2j-3}(p)\Delta_{2j+1}(p)}{\Delta^2_{2j-1}(p)}, \qquad j = 1, \ldots, \left[\frac{n-1}{2}\right], \tag{5.2}$$

and

$$b_{2j}b_{2j+1} = \frac{\Delta_{2j-2}(p)\Delta_{2j+2}(p)}{\Delta^2_{2j}(p)}, \qquad j = 0, 1, \ldots, \left[\frac{n-2}{2}\right], \tag{5.3}$$

where $\Delta_{-2}(p) \equiv 1$, and $[\alpha]$ denotes the maximal integer not exceeding α.

From the formulæ (5.2)–(5.3) and from Theorems 2.2 and 4.3 it is easy to obtain the following fact.

Theorem 5.1. *Let the matrix J_n be given in (5.1), and $n = 2l$. The characteristic polynomial p of the matrix J_n is generalized Hurwitz of type I if and only if*

$$b_0 > 0,\ b_1b_2 > 0,\ b_3b_4 > 0, \ldots,\ b_{n-3}b_{n-2} > 0. \tag{5.4}$$

The order $\varkappa$ of the polynomial p is equal to the number of negative terms in the sequence

$$b_0b_1,\ b_0b_1b_2b_3,\ b_0b_1b_2b_3b_4b_5, \ldots,\ b_0b_1\cdots b_{n-1}. \tag{5.5}$$

Conversely, let $\lambda_1, \ldots, \lambda_n$ be a sequence of complex numbers such that the polynomial $p(z) = \prod_{k=1}^n (z - \lambda_k)$ is generalized Hurwitz of type I of order $\varkappa$ and satisfies the inequalities (2.6). Then there exists a unique Schwarz matrix J_n of the form (5.1) with entries b_k satisfying (5.4) such that the number of negative terms in the sequence (5.5) is equal to $\varkappa$, and $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$.

Proof. Let p be the characteristic polynomial of the matrix J_n. It satisfies (2.6) by assumption. According to Theorem 4.3, it is generalized Hurwitz of type I if and only if $\Delta_{2i-1}(p) > 0$ for $i = 1, \ldots, l$. By (5.2), these inequalities are equivalent to (5.4). Furthermore, from (5.3) we have

$$\prod_{k=0}^{2i-1} b_k = \frac{\Delta_{2i}(p)}{\Delta_{2i-2}(p)}, \qquad i = 1, \ldots, l. \tag{5.6}$$

By Theorem 4.3, the order of the generalized Hurwitz polynomial p is equal to the number of sign changes in the sequence $\Delta_2(p),\ \Delta_4(p),\ \ldots,\ \Delta_{2l}(p)$. But from (5.6) we obtain that each sign change in this sequence corresponds to a negative number in the sequence (5.5).

Conversely, if the complex numbers $\lambda_1, \ldots, \lambda_n$ are such that the polynomial $p(z) = \prod_{k=1}^n (z - \lambda_k)$ is generalized Hurwitz of type I of order $\varkappa$ satisfying the inequalities (2.6), then by Theorems 2.2 and 4.3 and by formulæ (5.2)–(5.3), there exists a unique matrix J_n of the form (2.9) satisfying the inequalities (5.4) and with $\varkappa$ negative numbers in the sequence (5.5) such that its characteristic polynomial is p. ☐

Analogously, using formulæ (5.2)–(5.3) and Theorems 2.2, 4.3, and 4.12 one can easily establish the following theorems.

Theorem 5.2. *Let the matrix J_n be given in (5.1), and $n = 2l + 1$. The characteristic polynomial p of the matrix J_n is generalized Hurwitz of type I if and only if*

$$b_0b_1 > 0,\ b_2b_3 > 0,\ b_4b_5 > 0, \ldots,\ b_{n-3}b_{n-2} > 0. \tag{5.7}$$

The order $\varkappa$ of the polynomial p is equal to the number of negative terms in the sequence

$$b_0,\ b_0b_1b_2,\ b_0b_1b_2b_3b_4, \ldots,\ b_0b_1\cdots b_{n-1}. \tag{5.8}$$

Conversely, let $\lambda_1, \ldots, \lambda_n$ be a sequence of complex numbers such that the polynomial $p(z) = \prod_{k=1}^n (z - \lambda_k)$ is generalized Hurwitz of type I of order $\varkappa$ and satisfies the inequalities (2.6). Then there exists a unique Schwarz matrix J_n of the form (5.1) with entries b_k satisfying (5.7) such that the number of negative terms in the sequence (5.8) is equal to $\varkappa$, and $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$.

Theorem 5.3. *Let the matrix J_n be given in (5.1), and $n = 2l$. The characteristic polynomial p of the matrix J_n is almost generalized Hurwitz of type I if and only if*

$$b_0 b_1 > 0,\ b_2 b_3 > 0,\ b_4 b_5 > 0, \ldots,\ b_{n-2} b_{n-1} > 0. \tag{5.9}$$

The order $\varkappa$ of the polynomial p is equal to the number of negative terms in the sequence

$$b_0,\ b_0 b_1 b_2,\ b_0 b_1 b_2 b_3 b_4, \ldots,\ b_0 b_1 \cdots b_{n-2}. \tag{5.10}$$

Conversely, let $\lambda_1, \ldots, \lambda_n$ be a sequence of complex numbers such that the polynomial $p(z) = \prod_{k=1}^{n}(z - \lambda_k)$ is almost generalized Hurwitz of type I of order $\varkappa$ and satisfies the inequalities (2.6). Then there exists a unique Schwarz matrix J_n of the form (5.1) with entries b_k satisfying (5.9) such that the number of negative terms in the sequence (5.10) is equal to $\varkappa$, and $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$.

Theorem 5.4. *Let the matrix J_n be given in (5.1), and $n = 2l + 1$. The characteristic polynomialp of the matrix J_n is almost generalized Hurwitz of type I if and only if*

$$b_0 > 0,\ b_1 b_2 > 0,\ b_3 b_4 > 0, \ldots,\ b_{n-2} b_{n-1} > 0. \tag{5.11}$$

The order $\varkappa$ of the polynomial p is equal to the number of negative terms in the sequence

$$b_0 b_1,\ b_0 b_1 b_2 b_3,\ b_0 b_1 b_2 b_3 b_4 b_5, \ldots,\ b_0 b_1 \cdots b_{n-2}. \tag{5.12}$$

Conversely, let $\lambda_1, \ldots, \lambda_n$ be a sequence of complex numbers such that the polynomial $p(z) = \prod_{k=1}^{n}(z - \lambda_k)$ is almost generalized Hurwitz of type I of order $\varkappa$ and satisfies the inequalities (2.6). Then there exists a unique Schwarz matrix J_n of the form (5.1) with entries b_k satisfying (5.11) such that the number of negative terms in the sequence (5.12) is equal to $\varkappa$, and $\sigma(J_n) = \{\lambda_1, \ldots, \lambda_n\}$.

Remark 5.5. *It is also easy to prove and formulate an analogous theorems for (almost) generalized Hurwitz polynomials of type II. But it is not necessary, since if the characteristic polynomial of a matrix J_n is (almost) generalized Hurwitz of type II, then the characteristic polynomial of a matrix $-J_n$ is (almost) generalized Hurwitz of type I. But changing the sign of the matrix will change, in fact, just the sign of the entry b_0, since the characteristic polynomial of tridiagonal matrices depends on the products of the $(i, i+1)$th and $(i+1, i)$th entries rather than on these entries separately. So if we change their signs simultaneously, this does not change the characteristic polynomial [9, Chapter II]. Thus, if we have a matrix J_n of the form (5.1) such that $b_0 < 0$ and $b_0 b_1 < 0$, we should consider the matrix $-J_n$ and apply one of Theorems 5.1–5.4 (if any).*

6 Examples

In this section, we show how the results of the previous section can be used for certain sign patterns of the Schwarz matrix (5.1).

Consider the following matrix

$$S_n = \begin{bmatrix} -a & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ -c_1 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -c_2 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -c_k & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & c_{k-1} & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & c_{n-1} & 0 \end{bmatrix}, \tag{6.1}$$

where $a \in \mathbb{R} \backslash \{0\}$, and $c_k > 0$ for $k = 1, \ldots, n-1$.

Theorem 6.1. *Let $p(z)$ be the characteristic polynomial of the matrix S_n:*

$$p(z) = \det(zE_n - S_n).$$

- *If $n = 2l+1$ and $k = 2m+1$ or $n = 2l$ and $k = 2m$, then $p(z)$ is generalized Hurwitz of order $\varkappa = l - m$ of type I (type II) provided $a > 0$ (resp. $a < 0$).*
- *If $n = 2l+1$ and $k = 2m$ or $n = 2l$ and $k = 2m-1$, then $p(z)$ is almost generalized Hurwitz of order $\varkappa = l - m$ of type I (type II) provided $a > 0$ (resp. $a < 0$).*

Proof. Without loss of generality suppose that $a > 0$ (see Remark 5.5). From the conditions of the theorem and from the formulæ (2.8), we obtain that the characteristic polynomial p of the matrix S_n satisfies the inequalities

$$\Delta_{k+2+4i}(p) < 0, \qquad i = 0, 1, \dots, \left[\frac{n-k-2}{4}\right], \tag{6.2}$$

while all other Hurwitz determinants of p are positive. The statement of the theorem now follows from these inequalities and from Theorems 4.3 and 4.12. □

Converse theorem can also be established provided the given polynomial to satisfy the inequalities (6.2) while its other Hurwitz determinants are positive.

Theorem 6.2. *Let $\lambda_1, \dots, \lambda_n$ be a sequence of complex numbers such that the polynomial $p(z) = \prod\limits_{i=1}^{n}(z - \lambda_i)$ is a generalized Hurwitz polynomial of order $\varkappa$ of type I such that*

$$\Delta_{n-2\varkappa+2+4i}(p) < 0, \qquad i = 0, 1, \dots, \left[\frac{\varkappa-1}{2}\right], \tag{6.3}$$

and other $\Delta_j(p)$ are positive. Then there exists a unique Schwarz matrix S_n of the form (6.1) with $a > 0$ and $k = n - 2\varkappa$ such that $\sigma(S_n) = \{\lambda_1, \dots, \lambda_n\}$.

Proof. Indeed, by the conditions of the theorem, all the Hurwitz determinants of the polynomial p are nonzero, so according to Theorem 2.2, there exists a Schwarz matrix of the form (2.9) whose spectrum is $\{\lambda_1, \dots, \lambda_n\}$. But from the formulæ (2.8), from the inequalities (6.3) (see also (6.2)) and from the positivity of all other Hurwitz determinants of p, it follows that the sign pattern of the matrix S_n must be as in (6.1) with $k = n - 2\varkappa$. □

Analogously, one can prove the following theorem.

Theorem 6.3. *Let $\lambda_1, \dots, \lambda_n$ be a sequence of complex numbers such that the polynomial $p(z) = \prod\limits_{i=1}^{n}(z - \lambda_i)$ is almost generalized Hurwitz of order $\varkappa$ of type I such that*

$$\Delta_{n-2\varkappa+1+4i}(p) < 0, \qquad i = 0, 1, \dots, \left[\frac{\varkappa-1}{2}\right],$$

and other $\Delta_j(p)$ are positive. Then there exists a unique Schwarz matrix S_n of the form (6.1) with $a > 0$ and $k = n - 2\varkappa - 1$ such that $\sigma(S_n) = \{\lambda_1, \dots, \lambda_n\}$.

Remark 6.4. *Note that the results due to H. Wall and O. Holtz (Theorems 3.1 and 3.6) follow from Theorems 6.1, 6.2 and 6.3 for $k = 0$ and for $k = n - 1$ (see Remark 4.11).*

Finally, we show how to apply Theorem 6.1 to a more particular case. Consider the following matrix studied in [1]

$$J_n = \begin{bmatrix} a & 1 & 0 & \dots & 0 & 0 \\ -c_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & c_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & c_{n-1} & 0 \end{bmatrix}, \quad a > 0, c_j > 0. \tag{6.4}$$

By Theorem 6.1, the characteristic polynomial of this matrix is (almost) generalized Hurwitz polynomial of order $\varkappa = \left[\frac{n-1}{2}\right]$ of type II. In [1], there was posed the problem to find the condition on a sequence of complex number to be the spectrum of the matrix (6.4). The following theorem solves the direct and inverse problems for matrices of the form (6.4).

Theorem 6.5. *The eigenvalues λ_j of the matrix (6.4) are distributed in one of the following ways:*

1) $0 < -\lambda_1 < \lambda_2 < -\lambda_3 < \cdots < (-1)^n\lambda_{n-2}$, $\lambda_{n-1} = \overline{\lambda_n} \in \mathbb{C}$, *and* $\mathrm{Re}\lambda_n > 0$*;*

2) $0 < \lambda_1 \leqslant \lambda_2 < -\lambda_3 < \lambda_4 < -\lambda_5 < \cdots < (-1)^n\lambda_n$*;*

3) *for some k, $k = 1, \dots, \left[\frac{n-3}{2}\right]$,*
$0 < -\lambda_1 < \lambda_2 < \cdots < -\lambda_{2k-1} < \lambda_{2k} \leqslant \lambda_{2k+1} \leqslant \lambda_{2k+2} < \lambda_{2k+3} < \cdots < (-1)^{n-1}\lambda_{n-1} < (-1)^n\lambda_n$*;*

4) $0 < -\lambda_1 < \lambda_2 < -\lambda_3 < \cdots < (-1)^n\lambda_{n-2} \leqslant \lambda_{n-1} \leqslant \lambda_n$*, and* $(-1)^n\lambda_{n-2} < \lambda_{n-1}$ *if* $n = 2l+1$.

Conversely, let $\lambda_1, \dots, \lambda_n$ be a sequence of complex numbers satisfying one of the four conditions above, and $\sum\limits_{i=1}^{n} \lambda_i > 0$. Then there exists a unique matrix J_n of the form (6.4) such that $\sigma(J_n) = \{\lambda_1, \dots, \lambda_n\}$.

Proof. As we already mentioned, by Theorem 6.1, the characteristic polynomial p of the matrix (6.4) is generalized Hurwitz of order $\varkappa = \left[\frac{n-1}{2}\right]$ of type II (if $n = 2l+1$) or almost generalized Hurwitz of order $\varkappa = \left[\frac{n-1}{2}\right]$ of type II (if $n = 2l$). According to Definitions 4.10, 4.1 and 4.2, the eigenvalues of the matrix J_n are distributed in one of the four ways described in the statement of the theorem. Additionally, from the form of the matrix (6.4) it follows that $\sum\limits_{i=1}^{n} \lambda_i = a > 0$.

Conversely, let $\lambda_1, \dots, \lambda_n$ be a sequence of complex numbers satisfying one of the four conditions above, and $\sum\limits_{i=1}^{n} \lambda_i > 0$. Then the polynomial $p(z) = \prod\limits_{i=1}^{n}(z - \lambda_i)$ is generalized Hurwitz of order $\varkappa = \left[\frac{n-1}{2}\right]$ of type II (if $n = 2l+1$) or almost generalized Hurwitz of order $\varkappa = \left[\frac{n-1}{2}\right]$ of type II (if $n = 2l$) by Definitions 4.10, 4.1 and 4.2. It is left to prove that p satisfies the inequalities (2.6).

Let $n = 2l+1$. Since $q(z) := p(-z)$ is generalized Hurwitz of type I of order $\varkappa = \left[\frac{n-1}{2}\right] = l$ by Definition 4.2, we have

$$\Delta_2(q) > 0,\ \Delta_4(q) > 0,\ \dots, \Delta_{2l}(q) > 0. \tag{6.5}$$

and

$$l - 1 = V^F(1, \Delta_1(q), \Delta_3(q), \dots, \Delta_{2l+1}(q)).$$

But $\Delta_1(q) = -\sum\limits_{i=1}^{n}(-\lambda_i) > 0$, so $V^F(1, \Delta_1(q)) = 0$ and therefore we have

$$l - 1 = V^F(\Delta_1(q), \Delta_2(q), \dots, \Delta_{2l+1}(q)).$$

Now the Frobenius rule of sign (see comments to Theorem 4.3 on p. 8) requires all the determinants $\Delta_3(q)$, $\Delta_5(q)$, …, $\Delta_{2l+1}(q)$ to be nonzero and satisfying the inequalities

$$(-1)^{i-1}\Delta_{2i-1}(q) > 0, \qquad i = 1, \dots, l+1. \tag{6.6}$$

From the inequalities (6.5)–(6.6), Theorem 5.2, and the formulæ (2.8) we obtain that there exists a unique matrix of the form

$$\begin{bmatrix} -a & 1 & 0 & \dots & 0 & 0 \\ -c_1 & 0 & 1 & \dots & 0 & 0 \\ 0 & c_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & c_{n-1} & 0 \end{bmatrix}, \quad a > 0, c_j > 0.$$

whose characteristic polynomial is q. Now Remark 5.5 gives us the assertion of the theorem for $n = 2l + 1$. The case $n = 2l$ can be established analogously. $\square$

Note that the additional condition $\sum_{i=1}^{n} \lambda_i > 0$ is substantial for solution of the inverse problem for the matrix (6.4). If this number is negative, then the matrix must have another sign pattern. But if this number is zero, the inverse problem has no solution.

We finish by noting that using results of Section 5 one can find more examples of sign patterns of Schwarz matrices with (almost) generalized Hurwitz characteristic polynomials. At least, given a Schwarz matrix, one can always say if its characteristic polynomial is (almost) generalized Hurwitz or not.

Acknowledgment. The author thanks N. Bebiano and C. da Fonseca for helpful discussions.

References

[1] N. Bebiano and J. da Providência. Inverse problems for pseudo-Jacobi matrices: existence and uniqueness results. *Inverse Problems*, 27(2):025005, 2011, DOI:10.1088/0266-5611/27/2/025005.

[2] C. Chen and H. Chu. A matrix for evaluating Schwarz's form, *IEEE Trans. Automatic Control*, 11(2):303–305, 1966.

[3] B. Datta. A constructive method for finding the Schwarz form of a Hessenberg matrix, *IEEE Trans. Automatic Control*, 19(5):616–617, 1974.

[4] B. Datta. On the similarity between a matrix and its Routh canonical form, *IEEE Trans. Automatic Control*, 20(2):273–274, 1975.

[5] B. Datta. An inertia theorem for the Schwarz matrix, *IEEE Trans. Automatic Control*, 20(2):274, 1975.

[6] B. Datta. On the computation of Routh canonical form of a Hessenberg matrix, *IEEE Trans. Automatic Control*, 20(4):571–572, 1975.

[7] L. Elsner and D. Hershkowitz. On the spectra of close-to-Schwarz matrices, *Linear Algebra Appl.*, 363:81–88, 2003.

[8] G. Frobenius. Über das Trägheitsgesetz der quadratischen Formen. *Sitz.-Ber. Acad. Wiss. Phys.-Math. Klasse, Berlin*, 241–256; 407–431, 1953.

[9] Ф. Р. Гантмахер, М. Г. Крейн. *Осцилляционные матрицы и ядра и малые колебания механических систем.* (Russian), Москва-Ленинград, ГТТИ, 1941.

English transl.: *Oscillation matrices and kernels and small vibrations of mechanical systems.* AMS Chelsea Publishing, Providence, RI, revised edition, 2002. Translation based on the 1941 Russian original. Edited and with a preface by Alex Eremenko.

[10] F. R. Gantmacher. *The theory of matrices. Vol. 1.* Translated from the Russian by K. A. Hirsch. Reprint of the 1959 translation. AMS Chelsea Publishing, Providence, RI, 1998.

[11] F. R. Gantmacher. *The theory of matrices. Vol. 2.* Translated by K. A. Hirsch Chelsea Publishing Co., New York, 1959.

[12] O. Holtz. The inverse eigenvalue problem for symmetric anti-bidiagonal matrices, *Linear Algebra Appl.*, 408:268–274, 2005.

[13] O. Holtz and M. Tyaglov. Structured matrices, continued fractions, and root localization of polynomials. *arXiv: 0912.4703*, to appear in *SIAM Review.*

[14] N. Puri and C. Weygandt. Second method of Liapunov and Routh's canonical form, *J. Franklin Inst.*, 276(5):365–384, 1963.

[15] H. R. Schwarz. Ein Verfahren zur Stabilitätsfrage bei Matrizen-Eigenwertproblemen, *Zeitschrift für angew. Math. Phys.*, 7(6):473–500, 1956.

[16] M. Tyaglov. Generalized Hurwitz polynomials. *arXiv:1005.3032*, to appear in *Complex analysis and Operator Theory.*

[17] H. S. Wall. Polynomials whose zeros have negative real parts, *Amer. Math. Mothly*, 52(6):308–322, 1945.

[18] H. S. Wall. *Analytic Theory of Continued Fractions.* D. Van Nostrand Company, Inc., New York, 1948.